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ON THE INVARIANCE OF q-CONVEXITY AND HYPERCONVEXITY UNDER FINITE HOLOMORPHIC SURJECTIONS

NGUYEN VAN KHUE AND LE VAN THANH

ABSTRACT. In this note we have proved that 0-convexity and hyperconvexity are invariant under finite holomorphic surjections. Invariance of cohomological q-convexity for the case of finite dimension also has been established.

It is known [7] that Steinness is invariant under finite holomorphic surjections. In this note we investigate the invariance property for 0-convexity generally, for cohomological completeness, for cohomological q-convexity of finite dimension, and for hyperconvexity.

1. The invariance of 0**-convexity.** We recall that a complex space X is called q-convex if there exists an exhaustion function φ on X which is strictly q-pseudoconvex outside some compact $K \subset X$. It is known [2] that X is 0-convex if and only if there exists a proper holomorphic map θ from X onto a Stein space S such that $\theta_* \mathcal{O}_X \simeq \mathcal{O}_S$ and θ is biholomorphic outside some set of the form $\theta^{-1}A$, where A is a finite subset of S. The proper surjection θ: $X \to S$ with S is Stein, such that $\theta_* \mathcal{O}_X \simeq \mathcal{O}_S$ is said to be the *Remmert reduction* of X. By [2] if X is holomorphically convex then there exists a Remmert reduction. In this section we prove the following theorem.

1.1 THEOREM. *Let φ: $X \to Y$ be a finite holomorphic surjective map. Then Y is* 0*-convex if and only if X is.*

The proof of Theorem 1.1 is based on the following assertion, essentially as in [9].

1.2 ASSERTION. A complex space X is 0-convex if and only if $\dim H^1(X, \mathcal{S}) < \infty$ for every coherent ideal subsheaf $\mathcal{S} \subset \mathcal{O}_X$.

PROOF. The necessity follows from a theorem of Andreotti-Grauert [1].

Conversely, assume that $\dim H^1(X, \mathcal{S}) < \infty$ for every coherent ideal subsheaf $\mathcal{S}$ of $\mathcal{O}_X$. We have to prove that X is 0-convex.

First we show that X is holomorphically convex. Let $V = \{x_n\}_{n=1}^{\infty}$ be a discrete sequence in X and let J_V denote the ideal subsheaf of $\mathcal{O}_X$ associated to V. Consider the exact sequence:

$$0 \to J_V \to \mathcal{O}_X \xrightarrow{\eta} \tilde{\mathcal{O}}V \to 0. \tag{1}$$

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By hypothesis and by exactness of the cohomology sequence associated to (1) we get

$$\dim \mathcal{O}(V)/\operatorname{Im}\eta = \dim C^\infty/\operatorname{Im}\eta < \infty. \tag{2}$$

Let $l_\infty(V)$ denote the subspace of $\mathcal{O}(V)$ consisting of bounded functions on V. Then $\dim C^\infty/l_\infty(V) = \infty$. Thus by (2) it follows that $\operatorname{Im}\eta \setminus l_\infty(V) \neq \varnothing$. This implies that $\sup|f(x_n)| = \infty$ for some $f \in \mathcal{O}(X)$. Hence X is holomorphically convex.

Let θ: $X \to S$ be the Remmert reduction of X. To prove that X is 0-convex it suffices to show that θ: $X \setminus K \to S \setminus \theta(K)$ is injective for some compact set $K \subset X$. For a contradiction suppose there is a discrete set $V = \{x_n, y_n\}_{n=1}^\infty$ such that $\theta(x_n) = \theta(y_n)$ for every $n \geqslant 1$. For each n let $\sigma_n \in \mathcal{O}(V)$ be given by the formula

$$\sigma_n(x_n) = 1, \quad \sigma_n | V - x_n = 0.$$

Since $\theta_* \mathcal{O}_X \simeq \mathcal{O}_S$ it is easy to see that $\{\sigma_n \bmod \eta\}$ $(n = 1, 2, \ldots)$ is linearly independent in $\mathcal{O}(V)/\operatorname{Im}\eta$. This contradicts (2). Hence Assertion 1.2 is proved.

1.3 Assertion. Let θ: $X \to Y$ be an n-analytic covering and let Y be normal. Then for every coherent ideal subsheaf $\mathcal{S}$ of $\mathcal{O}_Y$ there exists a morphism Q: $\theta_*\theta^*\mathcal{S} \to \mathcal{S}$ such that $Qe = \mathrm{id}$, where e: $\mathcal{S} \to \theta_*\theta^*\mathcal{S}$ is the canonical injection.

Proof. Let $V \subset Y$ be the branch locus of θ and let $U \subset Y$ be a Stein open subset of Y on which there exists an exact sequence

$$\mathcal{O}_Y^q \to \mathcal{O}_Y^p \xrightarrow{\eta} \mathcal{S} \to 0. \tag{4}$$

Then the sequence

$$\theta^*\mathcal{O}_Y^q \to \theta^*\mathcal{O}_Y^p \xrightarrow{\tilde\eta} \theta^*\mathcal{S} \to 0 \tag{5}$$

is also exact. Note that $\theta^*\mathcal{O}_Y^m \simeq \mathcal{O}_X^m$ for every $m \geqslant 1$.

Consider $\sigma \in H^0(U, \theta_*\theta^*\mathcal{S}) = H^0(\theta^{-1}(U), \mathcal{S})$. Since $\theta^{-1}(U)$ is Stein [7] we can find $\beta \in H^0(\theta^{-1}(U), \theta_X^p)$ such that $\tilde\eta\beta = \sigma$. Since Y is normal, the formula $P_U(\beta)(z) = (1/n)\Sigma_{j=1}^n \beta(x_j)$ for $z \in U \setminus V$ where $\theta^{-1}(z) = \{x_1, x_2, \ldots, x_n\}$, defines an element $P_U(\beta) \in \mathcal{O}_Y^p(U)$. Put

$$Q_U(\sigma) = \eta P_U(\beta). \tag{6}$$

It is easy to see that $Q_U(\sigma)$ is independent of choice of $\beta \in H^0(\theta^{-1}(U), \mathcal{O}_X^p)$, $\tilde\eta\beta = \sigma$, and $Q_U(\sigma) = \sigma$ for all $\sigma \in H^0(U, \mathcal{S})$.

Assume now that

$$\mathcal{O}_Y^{q'} \to \mathcal{O}_Y^{p'} \to \mathcal{S} \to 0 \tag{7}$$

in another exact sequence on U. Then there exists a commutative diagram:

$$\begin{array}{ccccccc} \mathcal{O}_Y^q & \to & \mathcal{O}_Y^p & \xrightarrow{\eta} & \mathcal{S} & \to & 0 \\ \downarrow & & \downarrow & & \| & & \\ \mathcal{O}_Y^{q'} & \to & \mathcal{O}_Y^{p'} & \to & \mathcal{S} & \to & 0 \end{array} \tag{8}$$

By the commutativity of (8) it follows that Q_U is independent of choice of presentation. Hence $Q = \{Q_U\}$ defines a morphism Q: $\theta_*\theta^*\mathcal{S} \to \mathcal{S}$ such that $Qe = \mathrm{id}$. Assertion 1.3 is proved.

Now we are able to prove Theorem 1.1. Assume that Y is 0-convex. Since φ is proper it follows that X is holomorphically convex. Considering the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \theta_X \downarrow & & \downarrow \theta_Y \\ S_X & \xrightarrow{\tilde{\varphi}} & S_Y \end{array} \tag{9}$$

where θ_X and θ_Y are Remmert reductions of X and Y respectively, it is easy to see that θ_X is biholomorphic outside some compact set K in X. Hence X is 0-convex.

Conversely, assume that X is 0-convex. We prove that Y is also 0-convex.

(a) First we consider the case, where $\dim Y < \infty$.

We assume that the theorem has been proved for all complex spaces Y of dimension $< m$. Now assume that $\dim Y = m$. Consider the commutative diagram:

$$\begin{array}{ccc} (X \times_Y \tilde{Y})_{\text{red}} & \xrightarrow{\tilde{\varphi}} & \tilde{Y} \\ \downarrow \eta & & \downarrow \nu \\ X & \xrightarrow{\varphi} & Y \end{array} \tag{10}$$

of finite holomorphic surjective maps, where $\tilde{Y}$ is the normalization of Y. By the necessary condition already proved, $(X \times_Y \tilde{Y})_{\text{red}}$ is 0-convex. On the other hand, since $\tilde{\varphi}$ is finite and $\tilde{Y}$ normal it follows that $\tilde{\varphi}$ is a finite analytic n-covering for some n [**4**]. Thus by 1.2 and 1.3 we infer that $\tilde{Y}$ is 0-convex. To prove that Y is 0-convex by 1.2 it suffices to show that $\dim H^1(Y, \mathcal{S}) < \infty$ for every coherent ideal subsheaf $\mathcal{S}$ of $\mathcal{O}_Y$. Let $\tilde{\mathcal{O}}_Y$ denote the coherent analytic sheaf of germs of weakly holomorphic functions on Y [**4**]. Put $\mathcal{D} = \mathcal{O}_Y : \tilde{\mathcal{O}}_Y$. Note that $\nu_* \mathcal{O}_{\tilde{Y}} = \tilde{\mathcal{O}}_Y$ and $\operatorname{supp} \theta_Y / \mathcal{D} = N(Y)$ where $N(Y)$ denotes the nonnormal locus of Y. Let $\mathcal{V}$ be the coherent ideal subsheaf of $\mathcal{O}_{\tilde{Y}}$ which is the image of $\nu^* \mathcal{D}\mathcal{S} = \nu^{-1}(\mathcal{D}\mathcal{S}) \otimes_{\nu^{-1}\mathcal{O}_Y} \mathcal{O}_{\tilde{Y}}$ under multiplication. By using the definition of $\mathcal{D}$ it follows that $\nu_* \mathcal{V} \subset \mathcal{S}$ and since $\tilde{Y}$ is 0-convex and ν is finite we have [**4**]

$$\dim H^1(Y, \nu_*, \mathcal{V}) = \dim H^1(\tilde{Y}, \mathcal{V}) < \infty. \tag{11}$$

Since ν is biholomorphic outside $\nu^{-1}(N(Y))$ it follows that

$$\operatorname{supp} \mathcal{S}/\nu_* \mathcal{V} \subset N(Y). \tag{12}$$

Thus, using the induction hypothesis we get

$$\dim H^1(Y, \mathcal{S}/\nu_* \mathcal{V}) = \dim H^1(N(Y), \mathcal{S}/\nu_* \mathcal{V}) < \infty. \tag{13}$$

By (11) and (13) and by the exactness of the cohomology sequence associated to the exact sequence

$$0 \to \nu_* \mathcal{V} \to \mathcal{S} \to \mathcal{S}/\nu_* \mathcal{V} \to 0,$$

we infer that $\dim H^1(Y, \mathcal{S}) < \infty$.

(b) In the general case, let $Y = \bigcup_{j=1}^{\infty} V_j$, where V is an irreducible branch of Y for any $j \geqslant 1$. Since $\tilde{Y} = \coprod_1^{\infty} \tilde{V}_j$, by the 0-convexity of $\tilde{Y}$ it is easy to see that there exists j_0 such that $\tilde{V}_j$ is Stein for every $j > j_0$. Hence V_j is also Stein for every $j > j_0$.

Put

$$Y_0 = \bigcup_{j=1}^{j_0} V_j, \quad Y_k = Y_0 \cup \bigcup_{j=1}^{k} V_{j_0+j}.$$

By (a) Y_k is 0-convex for every $k \geqslant 0$.

If Y_0 is Stein then Y is Stein by [7]. Now we assume that Y_0 is 0-convex non-Stein. Thus Y_k is 0-convex non-Stein for every $k \geqslant 0$. Let θ_k: $Y_k \to S_k$ be the Remmert reduction of Y_k. Then we have the following diagram:

$$\begin{array}{ccccccc} Y_0 & \overset{i_0}{\to} & Y_1 & \overset{i_1}{\to} & Y_2 & \overset{i_2}{\to} & \cdots \\ \theta_0 \downarrow & & \downarrow \theta_1 & & \downarrow \theta_2 & & \\ S_0 & \underset{\tilde{i}_0}{\to} & S_1 & \underset{\tilde{i}_1}{\to} & S_2 & \underset{\tilde{i}_2}{\to} & \cdots \end{array}$$

Let A_k be a finite subset of S_k such that θ_k: $Y_k - \theta_k^{-1}(A_k) \to S_k - A_k$ is biholomorphic. Since Y_k non-Stein, $\theta_k^{-1}(y)$ is connected of positive dimension for every $y \in A_k$ [2]. Then, since $\bigcup_{j > j_0} V_j$ is Stein, and $\theta_k^{-1}(A_k)$ is compact connected of positive dimension, it follows that

(14) $$\theta_k^{-1}(A_k) = \theta_0^{-1}(A_0) \quad \text{for every } k \geqslant 0.$$

From (14) it is easy to see that there exists k_0 such that

(15) $$\tilde{i}_k \text{ is proper injective and } \theta_0^{-1}(A_0) \subset \operatorname{Int} Y_k \quad \text{for every } k = k_0.$$

Put $S = \varinjlim S_k$ and $\theta = \varinjlim \theta_k$: $Y \to S$. By (14)(15) we infer that θ is proper, $\theta \mid Y \setminus \theta_0^{-1}(A_0)$ is biholomorphic, and $\theta_* \mathcal{O}_Y = \mathcal{O}_S$. Moreover, since $S = \varinjlim \tilde{i}_k(S_k)$ where $\tilde{i}_k(S_k)$ are Stein closed subspaces of S, it follows that S is holomorphically separated and holomorphically convex and thereby S is Stein. Hence Y is 0-convex. This completes the proof of Theorem 1.1.

The following is an immediate consequence of Theorem 1.1.

1.4 COROLLARY. *A complex space X is* 0*-convex if and only if all its irreducible branches, except for finitely many which are* 0*-convex, are Stein.*

1.5 COROLLARY. *Let θ: $X \to Y$ be a proper holomorphic surjective map which is finite outside a compact set. Then X is* 0*-convex if and only if Y is.*

PROOF. Assume that Y is 0-convex. Considering the commutative diagram (9) it is easy to see that θ_X is finite outside a compact set. Hence by the Steinness of S_X we infer that X is 0-convex. Now assume that X is 0-convex. Consider the commutative diagram (16)

(16) $$\begin{array}{ccc} X & \overset{\theta}{\to} & Y \\ \theta_X \downarrow & \searrow^{\eta} & \uparrow \theta' \\ S_X & \underset{\beta}{\leftarrow} & X' \end{array}$$

in which X' is the Stein factorization of X for θ, and η, β are canonical maps and θ' is induced by θ. It is easy to check that θ' is finite, β is finite outside a compact set. This implies that X' is 0-convex. By Theorem 1.1 we infer that Y is 0-convex.

1.6 COROLLARY. *Let* θ: $X \to Y$ *be a proper holomorphic surjective map and* X *be* 0-*convex. Then* Y *is also* 0-*convex.*

PROOF. Considering the commutative diagrams (10) and (16) and by Theorem (1.1), without loss of generality we may assume that Y is normal. Hence Y is holomorphically convex. Consider the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \\ \theta_X \downarrow & & \downarrow \theta_Y \\ S_X & \xrightarrow[\tilde{\theta}]{} & S_Y \end{array}$$

Since θ_X is finite outside a compact set, and $\tilde{\theta}$ is finite, it follows that θ_Y is finite outside a compact set. Hence Y is 0-convex.

1.7 REMARK. Corollary (1.6) is not true for the holomorphically convex property [7].

2. The invariance of cohomological q-**completness.** A complex space X is called cohomologically q-complete (resp. cohomologically q-convex) if and only if $H^p(X, \mathscr{S}) = 0$ (resp. $\dim H^p(X, \mathscr{S}) < \infty$) for every coherent ideal subsheaf $\mathscr{S}$ of $\mathscr{O}_X$ and for every $p > q$.

In this section we prove the following theorem.

2.1 THEOREM. *Let* φ: $X \to Y$ *be a finite holomorphic surjective map. Then* X *is cohomologically q-complete if and only if* Y *is. If, moreover,* $\dim X < \infty$ *then* X *is cohomologically q-convex if and only if* Y *is.*

PROOF. Since φ is finite it follows that if Y is cohomologically q-complete (resp. cohomologically q-convex) then X is too.

As in the proof of Theorem 1.1(a) it follows that if $\dim X < \infty$ and X is cohomologically q-complete (resp. cohomologically q-convex), then so is Y.

Thus to find the proof of the theorem it suffices to prove the following

2.2 ASSERTION. Let $X = \bigcup_{k=1}^{\infty} X_k$, X_k is the union of all irreducible branches of X of dimension $\prec k$. If X_k is cohomologically q-complete for every $k \geqslant 1$, then X is also cohomologically q-complete.

PROOF. Let $\mathscr{S}$ be a coherent ideal subsheaf of $\mathscr{O}_X$ and $J_k = \mathscr{T}_{X_k}$—the ideal subsheaf of $\mathscr{O}_X$ associated to X_k. By η_k: $\mathscr{O}_X \to \tilde{\mathscr{O}}_{X_k}$ denotes the canonical map. Put $\mathscr{S}_k = \eta_k(\mathscr{S})$. Since any open set in X is contained in some X_k it follows that

$$\mathscr{S} = \varprojlim \{ \mathscr{S}_k, \omega_k^j \},$$

where ω_k^j: $\mathscr{S}_k \to \mathscr{S}_j$ is a canonical map.

Let $\mathscr{U}$ be a Stein open covering of X. By hypothesis we have

$$H^p(\mathscr{U}, \mathscr{S}_k) = H^p(X, \mathscr{S}_k) = H^p(X_k, \mathscr{S}_k) = 0 \tag{17}$$

for every $p > q$ and so

$$\text{(18)} \qquad \operatorname{Im}\{H^{p-1}(\mathscr{U}, \mathscr{S}_{k+1}) \to H^{p-1}(\mathscr{U}, \mathscr{S}_k)\} = H^{p-1}(\mathscr{U}, \mathscr{S}_k)$$

for every $p > q$ and $k \geqslant 1$.

Consider $\sigma \in Z^p(\mathscr{U}, \mathscr{S})$, $p > q$. By (17) for each $k \geqslant 1$ we find $\beta'_k \in C^{p-1}(\mathscr{U}, \mathscr{S}_k)$ such that $\delta^{p-1}\beta'_k = \eta_k\sigma$. Put $\beta_1 = \beta'_1$ and consider $\omega^1_2\beta'_2 - \beta_1$. Since $\delta^{p-1}(\omega^1_2\beta'_2 - \beta_1) = 0$, by (18) with $k = 1$ we find $\beta''_2 \in Z^{p-1}(\mathscr{U}, \mathscr{S}_2)$ such that

$$\omega^1_2(\beta''_2 - \beta'_2) + \beta_1 = \delta^{p-2}\gamma \quad \text{for some } \gamma \in C^{p-2}(\mathscr{U}, \mathscr{S}_1).$$

Since $\mathscr{U}$ is a Stein open covering, there exists $\tilde{\gamma} \in C^{p-2}(\mathscr{U}, \mathscr{S}_2)$ such that $\omega^1_2\tilde{\gamma} = \gamma$. Put

$$\beta_2 = -\beta''_2 + \beta'_2 + \delta^{p-2}\tilde{\gamma}.$$

Then $\delta^{p-1}\beta_2 = \eta_2\sigma$ and $\omega^1_2\beta_2 = \omega^1_2(\beta'_2 - \beta''_2) + \omega^1_2\delta^{p-1}\tilde{\gamma} = \omega^1_2(\beta'_2 - \beta''_2) + \delta^{p-1}\omega^1_2\tilde{\gamma} = \omega^1_2(\beta'_2 - \beta''_2) + \beta_1 + \omega^1_2(\beta''_2 - \beta'_2) = \beta_1$. Continuing this process we get a sequence $\{\beta_n\}$ such that for every $n \geqslant 1$:

$$\beta_n \in C^{p-1}(\mathscr{U}, \mathscr{S}_n), \quad \delta^{p-1}(\beta_n) = \eta_n(\sigma) \quad \text{and} \quad \omega^n_{n+1}\beta_{n+1} = \beta_n.$$

Thus $\beta = \{\beta_n\} \in C^{p-1}(\mathscr{U}, \mathscr{S})$ and $\delta^{p-1}\beta = \sigma$. Hence $H^p(X, \mathscr{S}) = 0$ and 2.2 is proved.

The following is an immediate consequence of Theorem 2.1.

2.3 COROLLARY. *X is cohomologically q-complete if and only if every irreducible branch of X is.*

3. The invariance of the hyperconvexity. We recall that a Stein space X is called hyperconvex (resp. strongly hyperconvex) if there exists a plurisubharmonic (resp. strictly plurisubharmonic) negative exhaustion function on X [**8**]. In this section the following theorem is proved.

3.1 THEOREM. *Let θ: $X \to Y$ be a finite holomorphic surjective map of finite-dimensional complex spaces. Then*:

(i) *If Y is strongly hyperconvex having a strictly plurisubharmonic negative exhaustion C^2-function, then X is strongly hyperconvex.*

(ii) *If Y is irreducible and X is strongly hyperconvex having a strictly plurisubharmonic negative exhaustion C^2-function, then Y is strongly hyperconvex.*

We need the following.

3.2 LEMMA. *If X is strongly hyperconvex and Y is normal, then so is Y.*

PROOF. Let ψ be a strictly plurisubharmonic negative exhaustion function of X. By the integer lemma [**4**] we infer that θ: $X \to Y$ is an analytic covering. Thus we can define a function φ on Y by the formula

$$\text{(19)} \qquad \varphi(y) = \operatorname{Tr}_\theta(\psi)(y) = \sum_{\theta x = y} \psi(x)$$

(the points of $\theta^{-1}(y)$ being counted with the right multiplicity).

Since $\psi < 0$ it follows that φ is an exhaustion function. First we prove that φ is plurisubharmonic. By a theorem of Fornaess and Narasimham **[5]** it suffices to show that $\varphi\sigma$ is subharmonic for any holomorphic map σ of unit disc $D \subset C$ into Y.

Given such a map σ: $D \to Y$, consider the commutative diagram:

$$\begin{array}{ccc} (D \times_Y X)_{\text{red}} & \stackrel{\tilde{\sigma}}{\to} & X \\ \tilde{\theta} \downarrow & & \downarrow \theta \\ D & \to & Y \end{array}$$

in which θ and $\tilde{\theta}$ are analytic coverings. It is easy to see that the branching order $O_{\tilde{\theta}}(x) = O_{\tilde{\theta}}(\sigma x)$ for any $x \in (D \times_Y X)_{\text{red}}$. Thus $(\text{Tr}_\theta \psi)\sigma = \text{Tr}_{\tilde{\theta}}(\psi\tilde{\sigma})$. Hence it remains to show that $\text{Tr}_{\tilde{\theta}}(\psi\tilde{\sigma})$ is subharmonic. The problem is local on D, whence, without loss of generality, we can assume that there exists an embedding e: $(D \times_Y X)_{\text{red}} \to C^n$ for some n. Then we have the commutative diagram:

$$\begin{array}{ccc} (D \times_Y X)_{\text{red}} & \stackrel{\tilde{e}=(\tilde{\theta},e)}{\to} & D \times \mathbf{C}^n \\ & \tilde{\theta} \searrow & \downarrow \tilde{\pi} \\ & & D \end{array}$$

in which $\tilde{\pi} \mid A$: $A \to D$, $A = \tilde{e}(D \times_Y X)_{\text{red}}$, is an analytic covering. Since

$$\text{Tr}_{\tilde{\theta}}(\psi\tilde{\sigma}) \circ \tilde{e}^{-1} |_A = \text{Tr}_{\tilde{\pi}}\left(\psi \circ \tilde{\sigma} \circ \tilde{e}^{-1} |_A\right),$$

the subharmonicity of $\text{Tr}_{\tilde{\theta}}(\psi\tilde{\sigma})$ follows from a lemma of **[5]**.

If σ is a C^2-function on a neighborhood V of a point $y_0 \in Y$ such that partial derivatives of order $\leqslant 2$ have sufficiently small absolute values, then $\psi + \sigma\theta$ is plurisubharmonic. Since $\text{Tr}_\theta(\psi) + \sigma = \text{Tr}_\theta(\psi + \sigma\theta)$ we infer that $\text{Tr}_\theta(\psi) + \sigma$ is plurisubharmonic. Thus $\text{Tr}_\theta \psi$ is strictly plurisubharmonic by definition. The lemma is proved.

3.3 LEMMA. *If Y is irreducible and $\tilde{Y}$ is strongly hyperconvex, then so is Y.*

PROOF. Since Y is irreducible, the normalization map ν: $\tilde{Y} \to Y$ is homeomorphic. Thus $\psi \circ \nu^{-1}$ is a continuous negative exhaustion function on Y, where ψ is that function on Y. Since for every holomorphic map σ: $D \to Y$ the map $\nu^{-1}\sigma$ is holomorphic, as in the proof of the Lemma 3.2 we infer that $\psi\nu^{-1}$ is strictly plurisubharmonic. Hence Y is strongly hyperconvex.

PROOF OF THEOREM 3.1. (i) Let φ be a strictly plurisubharmonic negative exhaustion C^2-function on Y. We can assume that Y is embedded in C^n for some n. It is known **[6]** that there exists a relatively compact Stein open covering $\{U_j\}$ of C^n of finite order and a C^∞-partition $\{\rho_j\}$ of unity subordinate to $\{U_j\}$ such that $|D^a\rho_j(x)| \leqslant C_\alpha$ for all α and for all j. Since $\theta^{-1}(U_j)$ is a relatively compact Stein open set, we may find a strictly plurisubharmonic nonnegative ψ_j C^∞-function on $\theta^{-1}(U_j)$. We set

$$\psi(x) = \sum_j \rho_j(\theta x)\psi_j(x) + \varphi(\theta x).$$

By calculating $\partial^2\psi/\partial z\partial\bar{z}$ (in the local coordinate of X) we conclude that in choosing ψ_j such that the absolute values of their partial derivatives of order $\leqslant 2$ is sufficiently small, $\psi(x)$ is a strictly plurisubharmonic negative exhaustion function of X. Hence X is strongly hyperconvex.

(ii) Considering the commutative diagram:

$$\begin{array}{ccc} (X \times_Y \tilde{Y})_{\text{red}} & \xrightarrow{\tilde{\theta}} & \tilde{Y} \\ \tilde{\nu}\downarrow & & \downarrow\nu \\ X & \xrightarrow{\theta} & Y \end{array}$$

of the finite surjective maps, by (i) and by Lemma 3.2 and 3.3 we get strong hyperconvexity of Y. The theorem is proved.

3.4 REMARK. In [3] Diederich and Fornaess have proved that every Stein bounded domain in C^n with C^2-boundary has a strictly plurisubharmonic negative exhaustion C^2-function.

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INSTITUT DE MATHÉMATIQUES, P. O. BOX 631, BO HO, HANOI, VIETNAM